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FURTHER CANONICAL METHODS IN THE SOLUTION

OF VARIABLE-COEFFICIENT LANCHESTER-TYPE

EQUATIONS OF MODERN WARFARE

by

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and

Gerald G. Brown

June 1977

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0. INTRODUCTION

In an earlier paper[24] we developed some elements of a mathematical theory for solving variable-coefficient Lanchester-type equations of modern warfare for combat between two homogeneous forces and introduced canonical hyperbolic-like Lanchester functions for constructing their solution. Unfortunately, with only these previous results one is limited to computing (however conveniently it may be done) force-level trajectories and cannot gain a real understanding of qualitative model behavior (e.g. force annihilation) without the excessive labor of extensive numerical computations (and only then for specific values of model parameters). Since our earlier work important mathematical discoveries have been made about the qualitative behavior of this combat model, and we wish to show how these new results allow one to parametrically analyze combat modelled by power attrition-rate coefficients (see Section 1 below) with somewhat the same facility as he can study F. W. LANCHESTER's[17] classic constant-coefficient model. In order to obtain this analysis capability, however, one must redefine the Lanchester-Clifford-Schläfli (or LCS) functions, which we introduced in reference 24.

In our earlier paper (see TAYLOR and BROWN[24]) we gave various examples of hyperbolic-like Lanchester functions (in particular, the LCS functions, which arise from power attrition-rate coefficients with "no offset"). Subsequent research by TAYLOR and COMSTOCK[27] has revealed, however, that these canonical LCS functions must be redefined to permit force-annihilation prediction from initial conditions without having to spend the time and effort to compute force-level trajectories. It then became obvious that the entire topic of representing the solution to such Lanchester-type equations in terms of general Lanchester functions (GLF) should be critically reexamined. Thus, the purpose of this paper is to present new general considerations for the selection of canonical Lanchester functions and then apply this theory to the special case of power attrition-rate coefficients with "no offset" (modelling, for example, weapon systems with the same maximum effective range) to obtain new LCS functions. With the availability of tabulations of these new LCS functions

(see Section 6 below), one can study this model almost as easily and thoroughly as Lanchester's classic constant-coefficient one.

These power Lanchester (i.e. LCS) functions are significant not only because they correspond to attrition-rate coefficients modelling a large class of combat situations of interest but also because they yield valuable information about other related canonical Lanchester functions, e.g. the offset power Lanchester functions (see Note 1 and Section 8 below). This information is, of course, equivalent to knowledge about model behavior (e.g. force annihilation). As a result of our work here one can parametrically analyze variable-coefficient Lanchester-type models for combat between two homogeneous forces with somewhat the same facility as the classic constant-coefficient one. Such models are important for developing insights into the dynamics of combat (see BONDER and HONIG[10], TAYLOR[22], and Section 1 below).

The organization of this paper is as follows. First, we present the variable-coefficient Lanchester-type model that we study in this paper. Next, we discuss the representation of the time history of the force levels for this model in terms of general Lanchester functions (GLF). We show that there are essentially only two kinds of GLF, (I) exponential-like GLF and (II) hyperbolic-like GLF, and that the former (I) provide essential force-annihilation-prediction information about the latter (II). Then we explain why we have chosen to use the hyperbolic-like GLF to construct the model's solution and why the power Lanchester (or LCS) functions introduced by Taylor and Brown[24] must be redefined. Next, we show how the analysis of, for example, the X force-level equation is simplified by transforming the independent variable t to normalize the battle's time scale by the intensity of combat. We then introduce our new definition of Lanchester-Clifford-Schläfli functions and show how they arise in solving the transformed X force-level equation. Availability of tabulations of these new LCS functions is discussed, and some uses of the tabulations are illustrated. Finally, insights gained into the dynamics of combat between two homogeneous forces from these developments are discussed.

1. Variable-Coefficient Lanchester-Type Equations of Modern Warfare

In this paper we consider the following idealized model for combat between two homogeneous forces (see Note 2)

$$dx/dt = -a(t)y, \qquad\qquad dy/dt = -b(t)x, \qquad (1)$$

with initial conditions

$$x(t=0) = x_0, \qquad\qquad \text{and} \qquad\qquad y(t=0) = y_0,$$

where $t = 0$ denotes the time at which the battle begins, $x(t)$ and $y(t)$ denote the numbers of X and Y at time t, and $a(t)$ and $b(t)$ denote time-dependent Lanchester attrition-rate coefficients. We will refer to (1) as variable-coefficient Lanchester-type equations of modern warfare in honor of the pioneering military operations research work of F. W. Lanchester[17] (see TAYLOR[21] and Taylor and Brown[24]). Other forms of Lanchester-type equations appear in the literature, but we will not consider these here (see DOLANSKY[14] and Taylor[21]). The Lanchester-type equations (1) yield the <u>X force-level equation</u>

$$d^2x/dt^2 - \{d \ln a(t)/dt\}dx/dt - a(t)b(t)x = 0, \qquad (2)$$

with initial conditions

$$x(t=0) = x_0, \qquad\qquad \text{and} \qquad\qquad \{[1/a(t)]dx/dt\}_{t=0} = -y_0.$$

Although combat between two military forces is a complex random process, such an idealized deterministic model of the combat attrition process is frequently employed to provide insights into the dynamics of combat (see, for example, BONDER and FARRELL[9], Bonder and Honig[10], TAYLOR and PARRY[28], or WEISS[29]). The reader may consider (1) to model combat in which both sides use aimed fire and target acquisition times are independent of the numbers of firers and targets (see Note 3). New operations research techniques (see, for example, Bonder and Farrell[9], and CLARK[12]) for forecasting temporal variations in fire effectiveness (caused by, for example, changes in force separation, combatant postures, target acquisition rates, etc.) have generated interest in such variable-coefficient combat formulations.

Without loss of generality, we may take $a(t)$ and $b(t)$ to be of the form

$$a(t) = k_a g(t), \qquad \text{and} \qquad b(t) = k_b h(t), \qquad (3)$$

where $g(t)$ and $h(t)$ denote the time-varying factors of $a(t)$ and $b(t)$ such that $a(t)/b(t) = k_a/k_b$ for $g(t) = h(t)$. In other words, k_a and k_b denote "scale" factors chosen so that the case of constant coefficients corresponds to $g(t) = h(t) =$

A large class of tactical situations of interest can be modelled with the following general power attrition-rate coefficients

$$a(t) = k_a(t+K_S)^\mu, \qquad \text{and.} \qquad b(t) = k_b(t+K_S+K_0)^\nu, \qquad (4)$$

where $K_S, K_0 \geq 0$. The modelling roles of K_S and K_0 are discussed in Taylor and Brown[24]. We will call K_S the starting parameter, since it allows us to model (with $\mu, \nu \geq 0$) battles which begin within the maximum effective ranges of the two systems. We will call K_0 the offset parameter, since it allows us to model (again, with $\mu, \nu \geq 0$) battles between weapon systems with different effective ranges. Restrictions that must be placed in μ and ν, which are not necessarily integers, are discussed below.

Let us take a few moments to motivate our above notation and further indicate possible applications of our results. Consider BONDER's[4,6] constant-speed attack on a static definsive position modelled by

$$dx/dt = -\alpha(r)y = -\alpha_0(1-r/R_\alpha)^\mu y, \qquad dy/dt = -\beta(r)x = -\beta_0(1-r/R_\beta)^\nu x, \qquad (5)$$

where $\mu, \nu \geq 0$ and R_α denotes the maximum effective range of the Y weapon system. Then the starting parameter and the offset parameter are given by

$$K_S = (R_\alpha-R_0)/v, \qquad \text{and} \qquad K_0 = (R_\beta-R_\alpha)/v, \qquad (6)$$

where R_0 denotes the battle's opening range and $v > 0$ denotes the constant attack speed. Hence, $K_0, K_S \geq 0 \Leftrightarrow R_\beta \geq R_\alpha \geq R_0$. By considering (6) and Figure 1, the reader should have no trouble in understanding our terminology for K_S and K_0. In the model (5) μ, for example, is used to model the range dependence of Y's attrition-rate

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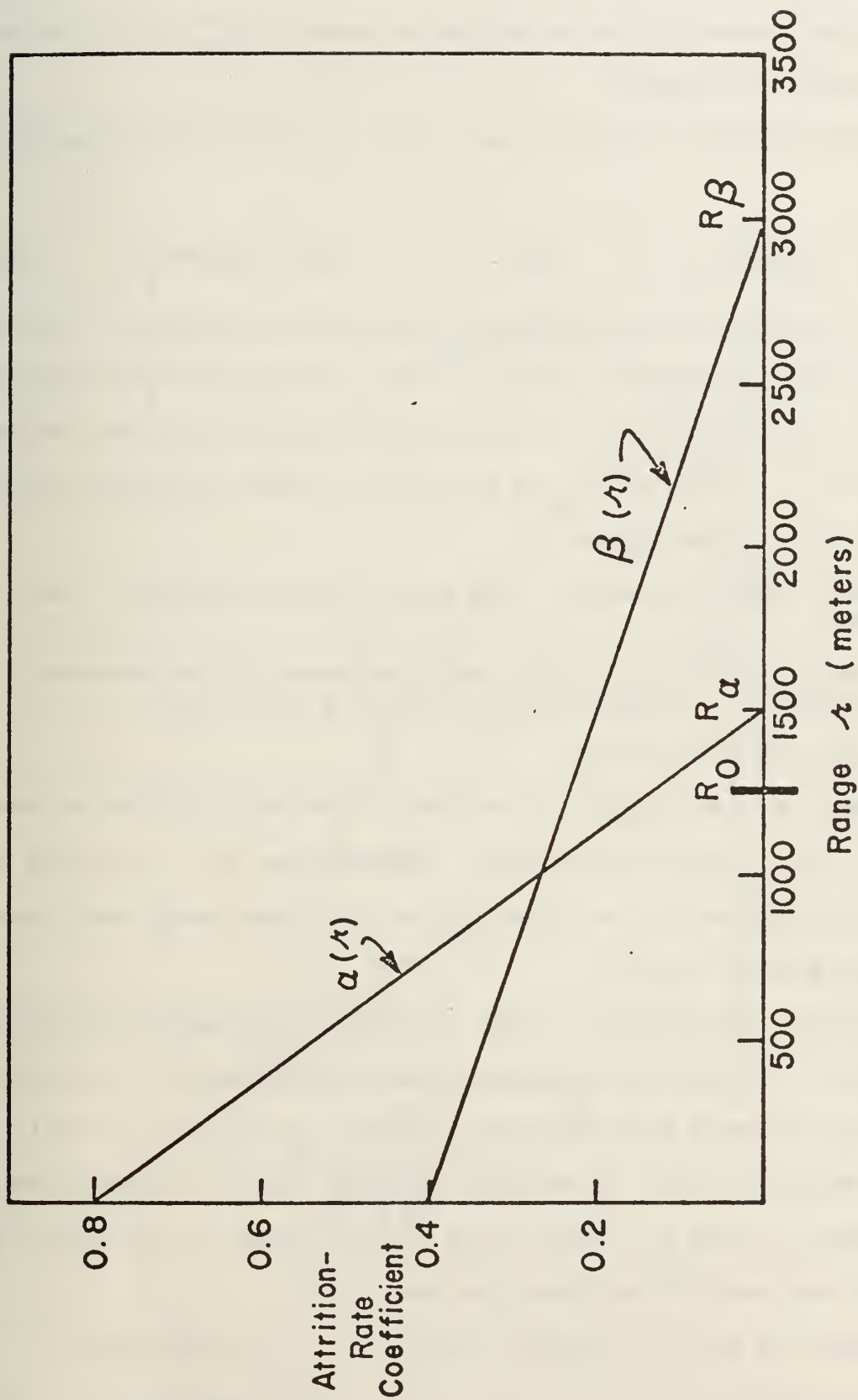
Figure 1. Explanation of offset parameter K_0 and starting parameter K_S for power attrition-rate coefficients modelling constant-speed attack. [Notes: 1. The maximum effective ranges of the two weapon systems are denoted as R_α and R_β. 2. The opening range of battle is denoted as R_0 and (as shown) $R_0 <$ minimum (R_α, R_β). 3. The offset parameter is given by $K_0 = (R_\beta - R_\alpha)/v$. 4. The starting parameter is given by $K_S = (R_\alpha - R_0)/v$.]

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coefficient (see Figure 2). Observing that range is related to time by $r(t) = R_0 - vt$, we readily see that the longest the battle can last is given by $t_{max} = R_0/v$, at which time zero force separation is reached.

When the offset parameter is equal to zero (i.e. $K_0 = 0$), then the coefficients (4) reduce to

$$a(t) = k_a(t+K_S)^\mu, \qquad \text{and} \qquad b(t) = k_b(t+K_S)^\nu. \qquad (7)$$

We will refer to (7) as power attrition-rate coefficients with "no offset." The purpose of this paper is to extend our previous results[24] and introduce new power Lanchester functions that allow more information to be more conveniently extracted from the model (1) with coefficients (7). Specifically, one would want to obtain information such as:

(Q1) Who will "win"? Be annihilated?

(Q2) How do force levels decrease over time and how many survivors will the winner have?

(Q3) How do changes in the initial force levels and weapon system parameters affect the outcome? Is concentration of forces a good tactic?

(Q4) How long will the battle last?

To conveniently answer questions (Q1), (Q3), and (Q4) a redefinition of the Lanchester-Clifford-Schläfli (or LCS) functions is required. Moreover, not only are results for the coefficients (7) of interest in their own right but they also provide much valuable information about the general case (4).

2. REPRESENTATION OF SOLUTION IN TERMS OF GENERAL LANCHESTER FUNCTIONS

In this section we discuss how to construct the solution to the X force-level equation (2) in terms of certain basic building blocks that we have chosen to call general Lanchester functions (GLF). We feel that these GLF should be chosen according to the guidelines shown in Table I. Special cases of these general considerations have been given by Taylor and Brown[24] and Taylor and Comstock[27].

Let us introduce the GLF $\underset{\sim}{x}^T = (x_1 \ x_2)$ and $\underset{\sim}{y}^T = (y_1 \ y_2)$ which satisfy

$$\dot{\underset{\sim}{x}} = ka(t)L\underset{\sim}{y}, \qquad\qquad \dot{\underset{\sim}{y}} = (1/k)b(t)L\underset{\sim}{x}, \qquad (8)$$

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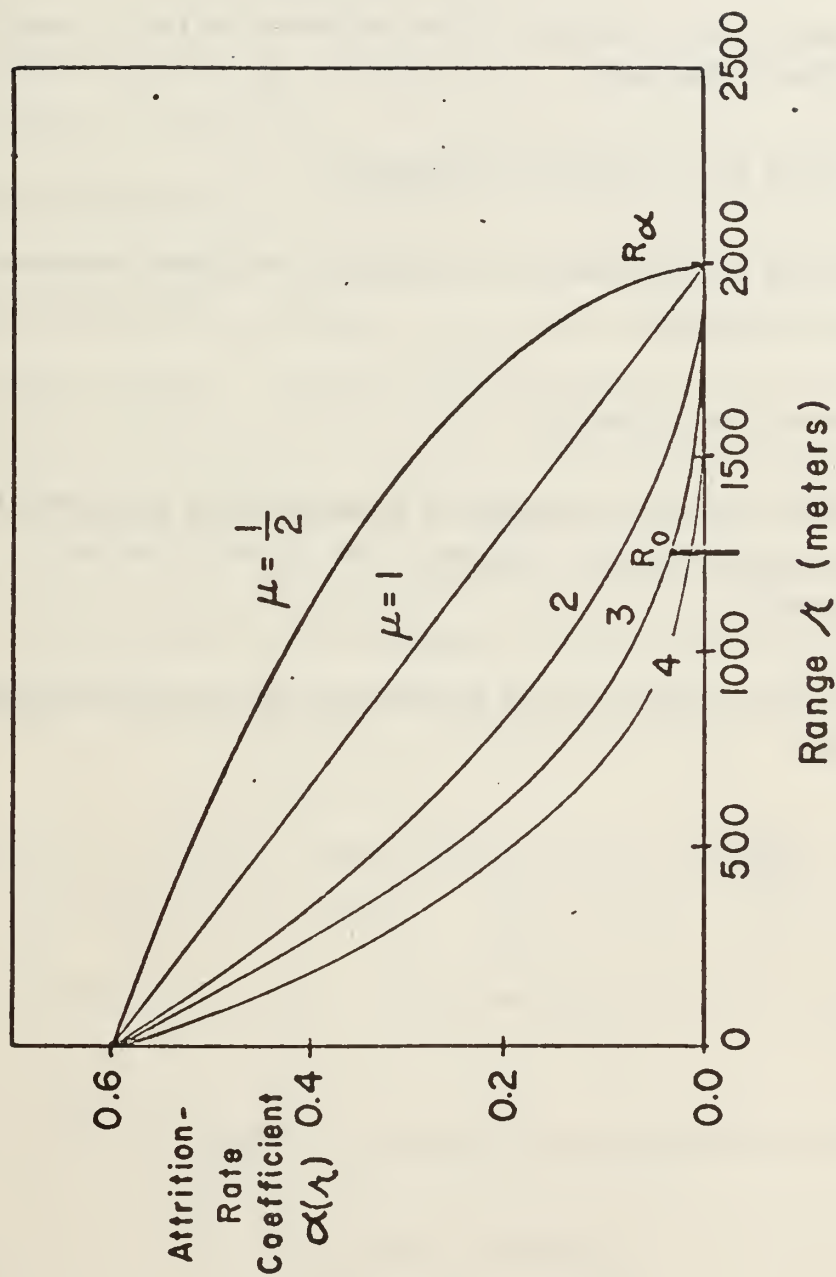
Figure 2. Dependence of the attrition-rate coefficient $\alpha(r)$ on the exponent μ with maximum effective range of the weapon system and kill capability at zero range held constant. [Notes: 1. The maximum effective range of the system is denoted as R_α = 2000 meters. 2. $\alpha(r=0) = \alpha_0$ = 0.6 X casualties/(unit time×number of Y units) denotes the Y force weapon system kill rate at zero force separation (range). 3. The opening range of battle is denoted as R_0 = 1250 meters and (as shown) $R_0 < R_\alpha$.]

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TABLE I. Requirements for General Lanchester Functions

(R1): They can be used to construct the solutions to the X and Y
       force-level equations.

(R2): They should be as "simple" as possible.

(R3): A given set of functions should apply to as large a class of
       battles as possible.

(R4): They should be nonnegative.

(R5): They should reduce to elementary transcendental functions in
       special cases such as a constant ratio of attrition-rate
       coefficients.

(R6): They should provide as much information as possible about model
       behavior.

where without further specification k may be any positive constant and L is any 2×2 square matrix such that $L^2 = I$. [The initial conditions for (8) are to be chosen so that the requirements of Table I are met and are discussed below.] Then $\underset{\sim}{x}$ satisfies the vector equation

$$\ddot{\underset{\sim}{x}} - \{d \ln a(t)/dt\}\dot{\underset{\sim}{x}} - a(t)b(t)\underset{\sim}{x} = 0. \qquad (9)$$

In other words, both x_1 and x_2 satisfy the X· force-level equation (2), and similarly y_1 and y_2 satisfy the Y force-level equation.

Let us now investigate all the possible forms for the above GLF. Intuitively, we would expect two possibilities (keeping the requirements of Table I in mind): exponential-like functions (one strictly increasing and the other strictly decreasing) and hyperbolic-like functions. [We are reminded of these two possibilities by the well-known constant-coefficient results.] Two such types of GLF appear in Taylor and Brown[24] (hyperbolic-like functions) and in Taylor and Comstock[27] (exponential-like functions). We will show that these are the only two possibilities if the requirements of Table I are to met and show the relationship between these two types of GLF.

It is readily shown that any 2×2 matrix such that $L^2 = I$ must take one of five forms.

LEMMA 1: If $L^2 = I$, then L must take one of the following five forms: (A) $\begin{pmatrix} \alpha & \beta \\ [1-\alpha^2]/\beta & -\alpha \end{pmatrix}$ with $\beta \neq 0$, (B) $\begin{pmatrix} 1 & 0 \\ \gamma & -1 \end{pmatrix}$, (C) $\begin{pmatrix} -1 & 0 \\ \gamma & 1 \end{pmatrix}$, (D) $L = I$, or (E) $L = -I$, where α, β, and γ are unrestricted with the exception that $\beta \neq 0$.

If in addition $L = L^T$ and $|L| = -1$, then L is an orthogonal matrix and must be of the form

$$L = \begin{pmatrix} \cos \phi & \sin \phi \\ \sin \phi & -\cos \phi \end{pmatrix}. \qquad (10)$$

It seems reasonable to give our Lanchester functions symmetry by requiring that $L = L^T$. We observe that the hyperbolic-like GLF of Taylor and Brown[24] correspond to

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$L_H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In order that (R4) of Table I be met for a constant ratio of attrition-rate coefficients, i.e.

$$a(t) = k_a h(t), \qquad \text{and} \qquad b(t) = k_b h(t), \tag{11}$$

we know (see Taylor and Brown[24]) that we should choose $k = \sqrt{k_b/k_a}$ so that

$$\dot{\underset{\sim}{x}} = \sqrt{k_b/k_a}\, a(t) L\underset{\sim}{y}, \qquad\qquad \dot{\underset{\sim}{y}} = \sqrt{k_a/k_b}\, b(t) L\underset{\sim}{x}. \tag{12}$$

The general solution to the X force-level equation (2) is given by

$$x(t) = \underset{\sim}{c}^T \underset{\sim}{x}. \tag{13}$$

Introducing the 2×2 matrix

$$S(t) = \begin{pmatrix} \underset{\sim}{x}^T(t) \\ \underset{\sim}{y}^T(t) L^T \end{pmatrix}, \tag{14}$$

we obtain from the initial conditions to (2) that

$$\underset{\sim}{c} = S^{-1}(t=0) \begin{pmatrix} \underset{\sim}{x}_0 \\ -\sqrt{k_a/k_b}\, \underset{\sim}{y}_0 \end{pmatrix}. \tag{15}$$

Observing that

$$\frac{d}{dt}\,|S(t)| = \begin{vmatrix} \dot{\underset{\sim}{x}}^T(t) \\ \underset{\sim}{y}^T(t) L^T \end{vmatrix} + \begin{vmatrix} \underset{\sim}{x}^T(t) \\ \dot{\underset{\sim}{y}}^T(t) L^T \end{vmatrix},$$

we readily see that $|S(t)| = \text{constant } \forall\, t\, \varepsilon\, (t_0, +\infty)$, where $|S|$ denotes the determinant of the square matrix S. Thus, we may take

$$|S(t)| = \text{constant}. \tag{16}$$

Let us also observe that

$$\sqrt{k_b/k_a}\, a(t) |S(t)| = W(x_1, x_2), \tag{17}$$

where $W(x_1, x_2)$ denotes the Wronskian of x_1 and x_2.

Now let us subject the fundamental system of solutions $\underset{\sim}{x}$ to linear transformation

$$\hat{\underset{\sim}{x}} = A\underset{\sim}{x}, \tag{18}$$

such that the form of the equations (12) remains invariant, i.e.

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$$\dot{\hat{\underset{\sim}{x}}} = \sqrt{k_b/k_a}\; a(t)\hat{L}\hat{\underset{\sim}{\chi}}, \qquad\qquad \dot{\hat{\underset{\sim}{\chi}}} = \sqrt{k_a/k_b}\; b(t)\hat{L}\hat{\underset{\sim}{x}}, \qquad (19)$$

where $\hat{L}$ again is such that $\hat{L}^2 = I$. If $\hat{L}$ is given, it follows that

$$\hat{\underset{\sim}{\chi}} = \hat{L}AL\underset{\sim}{\chi}. \qquad (20)$$

Furthermore

$$\hat{S}^T(t) = AS^T(t), \qquad (21)$$

so that

$$|\hat{S}(t)| = |A| \cdot |S(t)| = \text{constant}. \qquad (22)$$

We also observe that $W(\hat{x}_1, \hat{x}_2) = |A|W(x_1, x_2)$. Considering the quotient of the two general Lanchester X-functions (GLXF)

$$\eta(t) = x_1/x_2, \qquad (23)$$

we see that under the linear transformation (18) we have

$$d\hat{\eta}/dt = \{|A|/(a_{12}\eta + a_{22})^2\}d\eta/dt. \qquad (24)$$

We now show that the only possible GLF that satisfy the requirements of Table I (corresponding to $L = L^T$) are the exponential ones shown in Table II and the hyperbolic ones shown in Table III. According to Lemma 1 if $L^2 = I$, then L must take one of five forms. It is impossible to have $L = -I$ and satisfy (R4) of Table I (see Note 4). If $L = I$ and we try to specify the "simplest" initial conditions [i.e. specify initial conditions such that the GLF take the "simplest" form (satisfy (R2) of Table I)], we find that we may take L to have one of the three remaining forms (see Note 5). If we require that L be symmetric for simplicity [requirement (R2)], then L is an orthogonal matrix with the form (10). If in (10) we take $\cos\phi$ and $\sin\phi$ equal to -1, 0, or 1 in order that the GLF take the "simplest" form, we find that the only two distinct possibilities for L are L_E and L_H as given in Tables II and III (see argument given in Note 5). Thus, we have shown that if we wish to construct the solution (13) to (2) by using GLF with the properties given in Table I, there are essentially only two possibilities: the exponential-like GLF introduced by

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Table II. Exponential-Like General Lanchester Functions

1. $\underset{\sim E}{x}^T(t) = (E_X^+(t;Q*) \quad E_X^-(t;Q*)), \quad \underset{\sim E}{y}^T(t) = (E_Y^+(t;Q*) \quad E_Y^-(t;Q*))$

2. $\underset{\sim E}{x}^T(t=t_0) = .(1/Q* \quad 1), \quad \underset{\sim E}{y}^T(t=t_0) = (1 \quad Q*)$

3. $\qquad L_E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

4. $\qquad |S_E(t)| = -2$

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Table III. Hyperbolic-Like General Lanchester Functions

1. $\underset{\sim}{x}_H^T(t) = (S_X(t) \quad C_X(t)), \quad \underset{\sim}{y}_H^T(t) = (S_Y(t) \quad C_Y(t))$

2. $\underset{\sim}{x}_H^T(t=t_0) = (0 \quad 1), \quad \underset{\sim}{y}_H^T(t=t_0) = (0 \quad 1)$

3. $L_H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

4. $|S_H(t)| = -1$

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Taylor and Comstock[27] and the hyperbolic-like GLF introduced by Taylor and Brown[24].

We call the quantity $Q*$, which appears in the initial conditions for the exponential-like GLF, the <u>parity condition parameter</u>. It is chosen so that (<u>see</u> Note 6)

$$E_X^-(t;Q*), \quad E_Y^-(t;Q*) > 0 \quad \text{for all finite} \quad t \geq t_0. \tag{25}$$

It may be considered to be the enemy force equivalent of a friendly X force of unit strength. Taylor and Comstock[27] show how knowledge of the parity condition parameter allows one to predict force annihilation from initial conditions without explicitly computing force-level trajectories. We observe that the exponential-like GLF cannot be computed until one has solved the associated auxiliary parity-condition problem[27] (i.e. knows how to predict force-annihilation). For this reason and others (<u>see</u> Taylor and Comstock[27]), the exponential-like GLF are mainly of theoretical importance. Moreover, in the next paragraph we show how the exponential-like GLF provide valuable force-annihilation information about the hyperbolic-like GLF.

We now show that the limiting value of the quotient of the two hyperbolic-like GLXF, $\eta_H = S_X/C_X$, is equal to the reciprocal of the parity condition parameter, i.e. (30) holds. We know that the two types of GLF are related by a linear transformation

$$\underset{\sim}{x}_H = A\underset{\sim}{x}_E. \tag{26}$$

From (21) at $t = 0$, we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A \begin{pmatrix} 1/Q* & 1 \\ 1 & -Q* \end{pmatrix},$$

whence

$$A = \begin{pmatrix} 1/2 & -1/(2Q*) \\ Q*/2 & 1/2 \end{pmatrix}. \tag{27}$$

Considering (23) and (26), we see that

$$\eta_H(t) = \{a_{11}\eta_E(t;Q*)+a_{12}\}/\{a_{21}\eta_E(t;Q*)+a_{22}\}. \tag{28}$$

Recalling that $\eta_E(t;Q*) = E_X^+(t;Q*)/E_X^-(t;Q*)$ and $\lim_{t \to +\infty} E_X^-(t;Q*) = 0$, we see that

$$\lim_{t\to+\infty} \eta_E(t;Q^*) = +\infty, \tag{29}$$

so that by (27) and (28)

$$\lim_{t\to+\infty} \eta_H(t) = 1/Q^*. \tag{30}$$

This result (30) is highly significant because it leads to a computation method for determining Q^*. Moreover, in the future we will show how the LCS functions introduced in this paper (see Section 5 below) play a crucial role in such numerical determinations. Let us further note that (24) reads

$$d\eta_H/dt = \{2/(Q^*\eta_E+1)^2\}d\eta_E/dt, \tag{31}$$

so that both $\eta_E(t)$ and $\eta_H(t)$ are strictly increasing functions of t, since $d\eta_E/dt = 2\sqrt{k_b/k_a}\, a(t)/\{E_X^-(t;Q^*)\}^2$.

3. CONSIDERATIONS FOR THE CHOICE OF GENERAL LANCHESTER FUNCTIONS

In Table I we give the general requirements that we feel should be placed upon GLF. These requirements are motivated by the properties possessed by the functions (namely, the exponential and hyperbolic ones) that one uses to construct the solution to (1) in the constant-coefficient case.

We specify (R3) so that as few tabulations of GLF as possible will be required. Consequently, we specify the initial conditions for the GLF at $t_0 = \max(t_0^X, t_0^Y)$, where t_0^X denotes the largest finite singular point on the t-axis for the X force-level equation (2) (see reference 24). Thus, $a(t)$ and $b(t)$ are positive continuous functions $\forall t \in (t_0, +\infty)$. Since at most one of $x(t)$ and $y(t)$ can vanish in $[0, +\infty)$ (see Note 7), we have what the mathematician calls a nonoscilliatory solution to (1). In this case we can construct the solution to (2) out of nonnegative components and will find it convenient to do so.

As we have shown above, there are essentially only two types of GLF that satisfy the requirements of Table I: the exponential-like GLF of Table II and the hyperbolic-like GLF of Table III. We feel, however, that the hyperbolic-like functions are to be preferred for two reasons: (1) they apparently are more convenient for

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parametric studies in which one might, for example, want to vary initial force levels or some measure of relative fire effectiveness[24]; and (2) accurate values of the exponential-like GLF are, in general, difficult (in fact, essentially impossible for large values of t) to determine, since their initial conditions depend on the parity condition parameter Q* (see Note 8). In terms of the hyperbolic-like GLF, the solution to (2) is given by[24]

$$x(t) = x_0\{C_Y(t=0)C_X(t) - S_Y(t=0)S_X(t)\} - y_0\sqrt{k_a/k_b}\,\{C_X(t=0)S_X(t) - S_X(t=0)C_X(t)\}. \qquad (32)$$

We observe that for $t_0 < 0$, for example, $C_X(t=0) > 1$ and $S_X(t=0) > 0$ so that except for the quasi-autonomous case in which $a(t)/b(t) = $ constant (see Note 9), the solution (32) only simplifies when $t_0 = 0$ (see Theorem 1 of Taylor and Brown[24]) (see Note 10).

Unfortunately, the power Lanchester (or LCS) functions introduced by Taylor and Brown[24] were inappropriately defined to yield all the information sought about the combat model (1) with power attrition-rate coefficients (7). In particular, the time at which a side will be annihilated cannot be determined (without the explicit calculation of the entire force-level trajectories) from the initial conditions. Subsequent work by Taylor and Comstock has yielded a theory of force-annihilation prediction[2 The purpose of the paper at hand is to redefine the power Lanchester functions in light of these subsequent results. We also thought it important to present the general considerations behind this selection of canonical Lanchester functions.

Moreover, the form of the LCS functions is simplified and insight gained into the dynamics of combat by transforming the battle's time scale. Thus, certain transformations of variables may be desirable in the development of hyperbolic-like GLF, and the specifications of Table III should be interpreted as being "symbolic" and not taken literally.

4. A Transformation to Normalize the Battle's Time Scale
by the Intensity of Combat

Let $\int^t \ldots ds$ denote an indefinite integral, denote the relative effectiveness as $R(t)$, i.e.

$$R(t) = b(t)/a(t), \tag{33}$$

and let K be an arbitrary constant to be conveniently chosen. Then Theorem 2 of Taylor and Brown[24] may be stated as follows.

> THEOREM 1: A necessary and sufficient condition to be able to transform the X force-level equation (2) by a transformation of the independent variable t into a linear second order ordinary differential equation with constant coefficients is that
>
> $$[d\{\ell n\ R(t)\}/dt]/\sqrt{a(t)b(t)} = \text{CONSTANT},$$
>
> and then the desired substitution is given by
>
> $$\tau = K \int^t \sqrt{a(s)b(s)}\ ds.$$

We observe that Theorem 1 says that we can transform the X force-level equation to a constant coefficient one if and only if $d\{\ell n\ R(t)\}/d\tau = \text{CONSTANT}$. We also assume that the following condition holds.

> CONDITION (A): $\int_{t_0}^{t} a(s)ds$ and $\int_{t_0}^{t} b(s)ds$ are bounded for all finite
>
> $t \geq t_0$.

If Condition (A) is to hold, then for the power attrition-rate coefficients (7) we must have $\mu, \nu > -1$.

Motivated by both Theorem 1 and the well-known constant-coefficient results, we introduce the new independent variable τ defined by

$$\tau = \int_{t_0}^{t} \sqrt{a(s)b(s)}\ ds. \tag{34}$$

By Condition (A) and the Cauchy-Schwarz inequality for integrals (see p. 123 of BELLMAN[3]

the integral in (34) is well defined (i.e. bounded). The transformation (34) has an

inverse $t(\tau)$, since $d\tau/dt > 0 \; \forall \; t > t_0$. We also define

$$\tau_0 = \tau(t=0). \tag{35}$$

We observe that for $t_0 \leq 0$ we have $\tau_0 \geq 0$. Recalling the constant-coefficient results

we will call the quantity $\sqrt{a(t)b(t)}$ the "intensity of combat" (see also Taylor and

Parry[28]); since the larger it is, the more quickly the battle is moving towards ter-

mination. The average intensity of combat is given by $\overline{\sqrt{a(t)b(t)}} = (1/t) \int_0^t \sqrt{a(s)b(s)} \, ds$.

Then we have

$$\tau - \tau_0 = \{(1/t) \int_0^t \sqrt{a(s)b(s)} \, ds\}t = \overline{\sqrt{a(t)b(t)}} \; t. \tag{36}$$

The substitution (34) transforms (2) into

$$d^2x/d\tau^2 + (1/2)\{d \ln R(t)/d\tau\}dx/d\tau - x = 0, \tag{37}$$

with initial conditions

$$x(\tau=\tau_0) = x_0, \qquad \text{and} \qquad \{R^{1/2}(t)dx/d\tau\}_{\tau=\tau_0} = -y_0.$$

Theorem 1 tells us that unless (37) is a constant-coefficient equation, it is impossible

to transform the X force-level equation (2) into a constant-coefficient equation by

a transformation of the independent variable alone. Also, equation (37) is highly

significant because it clearly shows us that the course of combat depends on just two

weapon-system parameters: (1) $R(t) = b(t)/a(t)$, the relative fire effectiveness

(X to Y) of the two combatants, and (2) $I(t) = \sqrt{a(t)b(t)}$, the intensity of combat

(through equation (34), which relates $I(t)$ to τ). Both these parameters may vary

over time. In particular, from (37) we see that the nature of temporal variations in

relative fire effectiveness will have a significant effect upon the course of combat.

For the power attrition-rate coefficients with no offset (7), the transformed

X force-level equation (37) becomes

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$$d^2x/d\tau^2 + \{(2q-1)/\tau\}dx/d\tau - x = 0, \qquad (38)$$

with initial conditions

$$x(\tau=\tau_0) = x_0, \quad \text{and} \quad \{(\tau/2)^{2q-1}dx/d\tau\}_{\tau=\tau_0} = -y_0\sqrt{k_a/k_b}(\sqrt{k_a k_b}/(\mu+\nu+2))^{2q-1},$$

where $q = (\nu+1)/(\mu+\nu+2)$ and

$$\tau = \tau(t) = (2\sqrt{k_a k_b}/(\mu+\nu+2))(t+K_S)^{(\mu+\nu+2)/2}. \qquad (39)$$

Hence, $\tau_0 = (2\sqrt{k_a k_b}/(\mu+\nu+2))K_S^{(\mu+\nu+2)/2}$. Let us observe that $\forall \mu,\nu > -1$ we have $0 < q < 1$. Furthermore, $q > 1/2 \Leftrightarrow dR/dt > 0$, i.e. $R(t)$ is a strictly increasing function of time.

5. LANCHESTER-CLIFFORD-SCHLÄFLI FUNCTIONS

Consider the function $F_\alpha(x)$ defined by the power series

$$F_\alpha(x) = \Gamma(\alpha) \sum_{k=0}^{\infty} (x/2)^{2k}/\{k!\Gamma(k+\alpha)\}. \qquad (40)$$

For $\alpha \neq 0,-1,-2,\ldots$ the radius of convergence for $F_\alpha(x)$ is infinite by the ratio test for convergence of power series (see, for example, KNOPP[16]). Hence, $F_\alpha(z)$ is an entire function of the complex variable $z = x + iy$ with an essential singularity at the point at infinity. Now consider the function $H_\alpha(x)$ defined by the infinite series

$$H_\alpha(x) = \Gamma(\alpha) \sum_{k=0}^{\infty} (x/2)^{2(k+\alpha)}/\{k!\Gamma(k+\alpha+1)\}. \qquad (41)$$

Observing that

$$H_\alpha(x) = (1/\alpha)(x/2)^{2\alpha}F_{\alpha+1}(x), \qquad (42)$$

we see that for $\alpha > 0$ the infinite series (41) is uniformly convergent on compact subsets of the complex plane. From (42) we can readily deduce the recursive relation

$$F_\alpha(x) = F_{\alpha+1}(x) + \{(x/2)^2/[\alpha(\alpha+1)]\}F_{\alpha+2}(x). \qquad (43)$$

We will call the functions $F_\alpha(x)$ and $H_\alpha(x)$ Lanchester-Clifford-Schläfli functions (see Note 11). Other properties are readily deduced and are given in Table IV.

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Table IV. Properties of the LCS Functions $F_\alpha(x)$ and $H_\alpha(x)$.

1. $dF_\alpha/dx = (x/2)^{1-2\alpha} H_\alpha(x)$

2. $dH_\alpha/dx = (x/2)^{2\alpha-1} F_\alpha(x)$

3. $F_\alpha(x)F_{1-\alpha}(x) - H_\alpha(x)H_{1-\alpha}(x) = 1 \qquad \forall x$

   where α is not an integer (including zero)

4. $F_\alpha(x=0) = 1$

5. $H_\alpha(x=0) = 0 \qquad$ for $\qquad \alpha > 0$

6. $dF_\alpha/dx(x=0) = 0$

7. $\{(x/2)^{1-2\alpha} dH_\alpha/dx\}_{x=0} = 1$

8. $F_{1/2}(x) = \cosh x$

9. $H_{1/2}(x) = \sinh x$

The function $F_\alpha(x)$ satisfies the second order ordinary differential equation

$$d^2F_\alpha/dx^2 + \{(2\alpha-1)/x\}dF_\alpha/dx - F_\alpha = 0, \tag{44}$$

with initial conditions

$$F_\alpha(x=0) = 1, \qquad \text{and} \qquad dF_\alpha/dx(x=0) = 0,$$

while $H_\alpha(x)$ satisfies

$$d^2H_\alpha/dx^2 - \{(2\alpha-1)/x\}dH_\alpha/dx - H_\alpha = 0, \tag{45}$$

with initial conditions (for $\alpha > 0$)

$$H_\alpha(x=0) = 0, \qquad \text{and} \qquad \{(x/2)^{1-2\alpha}dH_\alpha/dx\}_{x=0} = 1.$$

Thus, $\{F_\alpha, H_{1-\alpha}\}$ is a fundamental system of solutions to

$$d^2F/dx^2 + \{(2\alpha-1)/x\}dF/dx - F = 0, \tag{46}$$

with Wronskian $W(F_\alpha, H_{1-\alpha}) = (x/2)^{1-2\alpha}$. Let us observe that (see Table III)

$$C_X(t) = F_q(\tau(t)), \quad S_X(t) = \{\sqrt{k_a k_b}/(\mu+\nu+2)\}^{2q-1}H_p(\tau(t)), \tag{47}$$

$$C_Y(t) = F_p(\tau(t)), \quad S_Y(t) = \{\sqrt{k_a k_b}/(\mu+\nu+2)\}^{1-2q}H_q(\tau(t)), \tag{48}$$

where $p = 1 - q$. If we define

$$T_\alpha(x) = H_{1-\alpha}(x)/F_\alpha(x), \tag{49}$$

then

$$\eta_H(t) = T_X(t) = S_X(t)/C_X(t) = \{\sqrt{k_a k_b}/(\mu+\nu+2)\}^{2q-1}H_p(\tau(t))/F_q(\tau(t)), \tag{50}$$

where $T_X(t)$ denotes a hyperbolic-like GLF. Observing that $\lim_{t\to+\infty} \tau(t) = +\infty$, we see that $T_\alpha(x)$ is a strictly increasing function of $x \ \forall x \in [0,+\infty)$ and

$$0 \le T_\alpha(x) < \Gamma(1-\alpha)/\Gamma(\alpha) \qquad \text{for} \qquad 0 \le x < +\infty, \tag{51}$$

with

$$\lim_{x\to+\infty} T_\alpha(x) = \Gamma(1-\alpha)/\Gamma(\alpha), \tag{52}$$

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since by the value of $Q*$ determined by Taylor and Comstock[27] for the power attrition-rate coefficients (7), denoted as $Q*(\mu, \nu, K_0=0)$, we have (see (30) and (50))

$$\lim_{t \to +\infty} T_X(t) = 1/Q*(\mu,\nu,K_0=0) = \{\sqrt{k_a k_b}/(\mu+\nu+2)\}^{2q-1} \Gamma(p)/\Gamma(q).$$

Comparing (38) and (46), we see that the solution to (38) is given by

$$x(t) = x_0\{F_p(\tau_0)F_q(\tau(t)) - H_q(\tau_0)H_p(\tau(t))\}$$

$$-y_0\sqrt{k_a/k_b}\{\sqrt{k_a k_b}/(\mu+\nu+2)\}^{2q-1}\{F_q(\tau_0)H_p(\tau(t)) - H_p(\tau_0)F_q(\tau(t))\}. \tag{53}$$

The time to annihilate X is determined by $x(t=t_X^a) = 0$ and thus

$$T_q(\tau(t_X^a)) = \{x_0 F_p(\tau_0)+y_0\sqrt{k_a/k_b}(\sqrt{k_a k_b}/(\mu+\nu+2))^{q-p}H_p(\tau_0)\}/$$

$$\{x_0 H_q(\tau_0)+y_0\sqrt{k_a/k_b}(\sqrt{k_a k_b}/(\mu+\nu+2))^{q-p}F_q(\tau_0)\}, \tag{54}$$

where $T_\alpha(\tau)$ is given by (49) and from (51)

$$0 \leq T_q(\tau) < \Gamma(p)/\Gamma(q). \tag{55}$$

For $t_0 = -K_S = 0$, (54) simplifies to

$$T_q(\tau(t_X^a)) = (x_0/y_0)\sqrt{k_b/k_a}(\sqrt{k_a k_b}/(\mu+\nu+2))^{p-q}. \tag{56}$$

From (54) and (55) we may deduce the following theorem:

THEOREM 2: Consider combat between two homogeneous forces described by (1) with power attrition-rate coefficients (7). Assume that these equations hold for all time and that Y "wins" when $x(t_f) = 0$ with $y(t_f) > 0$. Then Y will win if and only if

$$\Gamma(q)\{x_0 F_p(\tau_0)+y_0\sqrt{k_a/k_b}(\sqrt{k_a k_b}/(\mu+\nu+2))^{q-p}H_p(\tau_0)\} <$$

$$\Gamma(p)\{x_0 H_q(\tau_0)+y_0\sqrt{k_a/k_b}(\sqrt{k_a k_b}/(\mu+\nu+2))^{q-p}F_q(\tau_0)\}.$$

For $t_0 = 0$ (i.e. $K_S = 0$ and $\tau_0 = 0$), Y will win if and only if

$$x_0\Gamma(q) < y_0\sqrt{k_a/k_b}(\sqrt{k_a k_b}/(\mu+\nu+2))^{q-p}\Gamma(p).$$

6. TABULATIONS OF LCS FUNCTIONS

Tabulations of the Lanchester–Clifford–Schläfli functions are available in two of the authors' reports, also available from the National Technical Information Service (see references 25 and 26). These reports contain five-decimal-place tables of the hyperbolic-like LCs functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for values of the argument $x = 0.00(0.01)2.00(0.1)10.0$ and various values of the order α. The short table[25] contains tabulations for $\alpha = 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5, 3/7$, and $4/7$ corresponding to $\mu, \nu = 0, 1, 2, 3$; while the longer table[26] contains tabulations for $\alpha = 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5, 2/7, 3/7, 4/7, 5/7, 4/9, 5/9, 3/11, 5/11, 6/11, 8/11, 5/13, 8/13, 5/17, 12/17, 5/21$, and $16/21$ corresponding to $\mu, \nu = 0, 1/4, 1/2, 1, 1\ 1/2, 2, 3$. As we have seen above in Section 1 (see (5) and Figure 2), such values of μ and ν allow one to analyze, for example, a wide variety of range capabilities for weapon systems in Bonder's[4,6] constant-speed attack model (5). These tables have been calculated by the recursive methods given in Section 8 of Taylor and Brown[24].

A representative tabulation of the hyperbolic-like LCS functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 3/5$, similar to those that appear in references 25 and 26, is given in Tables V and VI. The values of the argument x are the same as those used for the tabulation of the hyperbolic functions by ABRAMOWITZ and STEGUN[1]. We observe from Table VI and (52) that the limiting value of $T_\alpha(x)$ as $x \to +\infty$ (here $\alpha = 3/5$) is quickly reached, with three-decimal-place agreement by $x = 4.5$.

7. NUMERICAL EXAMPLES

In this section we examine a couple of numerical examples to show some of the insights that may be gained into the dynamics of combat between two homogeneous forces from our new results. As in references 21 and 24, we consider S. Bonder's[4,6] model (5) for the constant-speed attack against a static defensive position. We will focus on the new results of this paper [in particular, the prediction of battle outcome from initial conditions without explicitly computing the force-level trajectories (cf. questions (Q1) and (Q4) of Section 1)]. From the input data given in Table VII, we

x	$F_{3/5}(x)$	$H_{2/5}(x)$	$T_{3/5}(x)$
0.00	1.00000	0.00000	0.00000
0.01	1.00007	0.03207	0.03607
0.02	1.00017	0.06287	0.06279
0.03	1.00038	0.08687	0.08684
0.04	1.00067	0.10937	0.10929
0.05	1.00104	0.13076	0.13063
0.06	1.00150	0.15133	0.15113
0.07	1.00204	0.17122	0.17088
0.08	1.00267	0.19058	0.19008
0.09	1.00338	0.20891	0.20877
0.10	1.00417	0.22798	0.22703
0.11	1.00505	0.24613	0.24490
0.12	1.00601	0.26398	0.26241
0.13	1.00705	0.28157	0.27959
0.14	1.00818	0.29891	0.29648
0.15	1.00939	0.31603	0.31309
0.16	1.01069	0.33296	0.32944
0.17	1.01207	0.34971	0.34554
0.18	1.01353	0.36631	0.36141
0.19	1.01508	0.38275	0.37706
0.20	1.01672	0.39906	0.39250
0.21	1.01844	0.41525	0.40773
0.22	1.02025	0.43129	0.42277
0.23	1.02213	0.44729	0.43761
0.24	1.02411	0.46317	0.45227
0.25	1.02617	0.47897	0.46675
0.26	1.02832	0.49468	0.48106
0.27	1.03055	0.51030	0.49520
0.28	1.03287	0.52590	0.50917
0.29	1.03527	0.54142	0.52298
0.30	1.03776	0.55689	0.53662
0.31	1.04034	0.57231	0.55011
0.32	1.04301	0.58769	0.56345
0.33	1.04576	0.60303	0.57664
0.34	1.04860	0.61834	0.58968
0.35	1.05153	0.63362	0.60257
0.36	1.05455	0.64887	0.61531
0.37	1.05765	0.66411	0.62791
0.38	1.06085	0.67933	0.64037
0.39	1.06413	0.69454	0.65268
0.40	1.06750	0.70974	0.66486
0.41	1.07097	0.72494	0.67690
0.42	1.07452	0.74013	0.68880
0.43	1.07816	0.75532	0.70057
0.44	1.08189	0.77052	0.71220
0.45	1.08572	0.78573	0.72370
0.46	1.08963	0.80095	0.73506
0.47	1.09364	0.81618	0.74630
0.48	1.09774	0.83143	0.75740
0.49	1.10193	0.84670	0.76838
0.50	1.10622	0.86199	0.77922

x	$F_{3/5}(x)$	$H_{2/5}(x)$	$T_{3/5}(x)$
0.50	1.10622	0.86199	0.77922
0.51	1.11060	0.87731	0.78984
0.52	1.11507	0.89265	0.80054
0.53	1.11963	0.90803	0.81104
0.54	1.12429	0.92343	0.82134
0.55	1.12905	0.93887	0.83156
0.56	1.13390	0.95435	0.84166
0.57	1.13885	0.96987	0.85163
0.58	1.14389	0.98544	0.86148
0.59	1.14903	1.00104	0.87121
0.60	1.15427	1.01670	0.88081
0.61	1.15960	1.03240	0.89030
0.62	1.16504	1.04815	0.89968
0.63	1.17057	1.06396	0.90893
0.64	1.17620	1.07983	0.91807
0.65	1.18193	1.09575	0.92709
0.66	1.18776	1.11176	0.93599
0.67	1.19370	1.12784	0.94478
0.68	1.19973	1.14389	0.95346
0.69	1.20587	1.16006	0.96202
0.70	1.21211	1.17632	0.97048
0.71	1.21845	1.19264	0.97882
0.72	1.22489	1.20903	0.98705
0.73	1.23145	1.22550	0.99517
0.74	1.23810	1.24204	1.00318
0.75	1.24486	1.25866	1.01109
0.76	1.25173	1.27537	1.01889
0.77	1.25870	1.29216	1.02658
0.78	1.26579	1.30903	1.03416
0.79	1.27298	1.32599	1.04165
0.80	1.28028	1.34304	1.04903
0.81	1.28769	1.36019	1.05630
0.82	1.29521	1.37742	1.06348
0.83	1.30284	1.39475	1.07055
0.84	1.31058	1.41218	1.07753
0.85	1.31843	1.42971	1.08440
0.86	1.32640	1.44734	1.09118
0.87	1.33448	1.46508	1.09786
0.88	1.34268	1.48292	1.10445
0.89	1.35099	1.50086	1.11094
0.90	1.35942	1.51892	1.11733
0.91	1.36797	1.53709	1.12363
0.92	1.37663	1.55537	1.12984
0.93	1.38541	1.57374	1.13596
0.94	1.39431	1.59229	1.14199
0.95	1.40333	1.61092	1.14792
0.96	1.41248	1.62968	1.15377
0.97	1.42174	1.64855	1.15953
0.98	1.43113	1.66752	1.16521
0.99	1.44064	1.68670	1.17080
1.00	1.45028	1.70597	1.17630

x	$F_{3/5}(x)$	$H_{2/5}(x)$	$T_{3/5}(x)$
1.00	1.45028	1.70597	1.17630
1.01	1.46004	1.72536	1.18472
1.02	1.46993	1.74539	1.18706
1.03	1.47995	1.76456	1.19231
1.04	1.49009	1.78436	1.19748
1.05	1.50037	1.80431	1.20258
1.06	1.51077	1.82439	1.20759
1.07	1.52111	1.84461	1.21253
1.08	1.53198	1.86500	1.21738
1.09	1.54278	1.88553	1.22217
1.10	1.55371	1.90621	1.22687
1.11	1.56479	1.92704	1.23150
1.12	1.57599	1.94802	1.23606
1.13	1.58734	1.96917	1.24055
1.14	1.59882	1.99047	1.24496
1.15	1.61044	2.01193	1.24930
1.16	1.62221	2.03356	1.25357
1.17	1.63411	2.05534	1.25778
1.18	1.64616	2.07731	1.26191
1.19	1.65836	2.09944	1.26598
1.20	1.67069	2.12175	1.26998
1.21	1.68318	2.14423	1.27392
1.22	1.69581	2.16688	1.27779
1.23	1.70859	2.18972	1.28159
1.24	1.72152	2.21274	1.28534
1.25	1.73460	2.23594	1.28902
1.26	1.74784	2.25932	1.29264
1.27	1.76123	2.28290	1.29620
1.28	1.77477	2.30667	1.29970
1.29	1.78847	2.33063	1.30314
1.30	1.80233	2.35479	1.30653
1.31	1.81634	2.37914	1.30985
1.32	1.83052	2.40369	1.31312
1.33	1.84485	2.42845	1.31634
1.34	1.85935	2.45342	1.31950
1.35	1.87402	2.47859	1.32261
1.36	1.88885	2.50395	1.32566
1.37	1.90384	2.52956	1.32866
1.38	1.91901	2.55537	1.33161
1.39	1.93434	2.58140	1.33451
1.40	1.94985	2.60765	1.33736
1.41	1.96553	2.63412	1.34016
1.42	1.98138	2.66082	1.34291
1.43	1.99741	2.68775	1.34562
1.44	2.01361	2.71490	1.34828
1.45	2.03000	2.74229	1.35089
1.46	2.04656	2.76992	1.35345
1.47	2.06331	2.79779	1.35597
1.48	2.08024	2.82589	1.35845
1.49	2.09735	2.85425	1.36088
1.50	2.11465	2.88285	1.36327

Table V. Lanchester-Clifford-Schläfli Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 3/5$ and x from 0.00 to 1.50.

α = 3/5

x	$F_{3/5}(x)$	$H_{2/5}(x)$	$T_{3/5}(x)$
1.50	2.11465	2.88285	1.36327
1.51	2.13214	2.91169	1.36562
1.52	2.14985	2.94080	1.36792
1.53	2.16770	2.97015	1.37019
1.54	2.18576	2.99977	1.37241
1.55	2.20402	3.02965	1.37460
1.56	2.22248	3.05979	1.37675
1.57	2.24118	3.09020	1.37886
1.58	2.25998	3.12088	1.38093
1.59	2.27904	3.15183	1.38296
1.60	2.29830	3.18306	1.38496
1.61	2.31777	3.21456	1.38692
1.62	2.33747	3.24635	1.38885
1.63	2.35733	3.27843	1.39074
1.64	2.37742	3.31079	1.39260
1.65	2.39773	3.34344	1.39442
1.66	2.41825	3.37639	1.39621
1.67	2.43899	3.40964	1.39797
1.68	2.45995	3.44319	1.39970
1.69	2.48113	3.47704	1.40139
1.70	2.50253	3.51120	1.40306
1.71	2.52416	3.54567	1.40469
1.72	2.54601	3.58046	1.40630
1.73	2.56810	3.61555	1.40787
1.74	2.59041	3.65098	1.40942
1.75	2.61296	3.68672	1.41094
1.76	2.63574	3.72280	1.41243
1.77	2.65876	3.75920	1.41389
1.78	2.68202	3.79594	1.41533
1.79	2.70553	3.83302	1.41674
1.80	2.72927	3.87044	1.41812
1.81	2.75327	3.90820	1.41948
1.82	2.77751	3.94632	1.42081
1.83	2.80205	3.98478	1.42212
1.84	2.82675	4.02361	1.42341
1.85	2.85115	4.06279	1.42467
1.86	2.87701	4.10234	1.42590
1.87	2.90253	4.14225	1.42712
1.88	2.92831	4.18253	1.42831
1.89	2.95436	4.22320	1.42948
1.90	2.98067	4.26423	1.43063
1.91	3.00726	4.30564	1.43175
1.92	3.03412	4.34747	1.43286
1.93	3.06125	4.38967	1.43394
1.94	3.08866	4.43226	1.43501
1.95	3.11636	4.47526	1.43605
1.96	3.14434	4.51865	1.43708
1.97	3.17259	4.56246	1.43809
1.98	3.20118	4.60668	1.43907
1.99	3.22998	4.65131	1.44004
2.00	3.25912	4.69636	1.44094

x	$F_{3/5}(x)$	$H_{2/5}(x)$	$T_{3/5}(x)$
2.0	3.25912	4.69636	1.44099
2.1	3.50725	5.17096	1.44957
2.2	3.90799	5.66257	1.45665
2.3	4.28459	6.26257	1.46065
2.4	4.70063	6.89726	1.46731
2.5	5.16006	7.59185	1.47127
2.6	5.66727	8.35659	1.47453
2.7	6.27710	9.19878	1.47722
2.8	6.84489	10.12649	1.47942
2.9	7.52655	11.14858	1.48123
3.0	8.27860	12.27485	1.48272
3.1	9.10823	13.51608	1.48394
3.2	10.02339	14.88417	1.48494
3.3	11.03685	16.39224	1.48577
3.4	12.14631	18.05476	1.48644
3.5	13.37444	19.88770	1.48699
3.6	14.77006	21.90868	1.48745
3.7	16.23317	24.13713	1.48782
3.8	17.87117	26.59442	1.48811
3.9	19.68890	29.30442	1.48837
4.0	21.63389	32.29303	1.48858
4.1	23.90546	35.59546	1.48874
4.2	26.34495	39.22452	1.48888
4.3	29.03689	43.22081	1.48899
4.4	32.00427	47.65715	1.48909
4.5	35.47878	52.53584	1.48916
4.6	38.89107	57.91754	1.48924
4.7	42.87709	63.85430	1.48928
4.8	47.27712	70.42877	1.48932
4.9	52.12255	77.42878	1.48935
5.0	57.47356	85.59991	1.48938
5.1	63.89728	94.39423	1.48940
5.2	69.89096	104.09703	1.48942
5.3	77.00779	114.80242	1.48944
5.4	85.00753	126.61432	1.48945
5.5	93.75716	139.64742	1.48946
5.6	103.64170	154.02832	1.48947
5.7	114.06960	169.89679	1.48947
5.8	125.80052	187.40709	1.48948
5.9	138.79270	206.72954	1.48948
6.0	153.10773	228.05212	1.48949

x	$F_{3/5}(x)$	$H_{2/5}(x)$	$T_{3/5}(x)$
6.0	153.10773	228.05212	1.48949
6.1	168.90493	251.58239	1.48949
6.2	186.33813	277.54946	1.48949
6.3	205.57712	306.20625	1.48950
6.4	226.80937	337.83198	1.48950
6.5	250.24183	372.73492	1.48950
6.6	276.10298	411.25538	1.48950
6.7	304.64504	453.27607	1.48950
6.8	336.46649	500.69077	1.48950
6.9	370.91477	552.47842	1.48950
7.0	409.28931	609.63759	1.48950
7.1	451.64484	672.72646	1.48950
7.2	498.39511	742.30131	1.48950
7.3	549.99691	819.22253	1.48950
7.4	606.95457	904.06134	1.48950
7.5	669.82494	997.70714	1.48950
7.6	739.22282	1101.07568	1.48950
7.7	815.82701	1215.13807	1.48950
7.8	900.38700	1341.10666	1.48950
7.9	993.73032	1480.16606	1.48950
8.0	1096.77069	1633.64526	1.48950
8.1	1210.90868	1803.01095	1.48950
8.2	1336.08100	1990.10239	1.48950
8.3	1474.69908	2194.50162	1.48950
8.4	1627.72303	2424.50162	1.48951
8.5	1796.65455	2676.12602	1.48951
8.6	1983.14952	2953.91126	1.48951
8.7	2189.30633	3260.58073	1.48951
8.8	2416.27389	3599.14166	1.48951
8.9	2667.27158	3972.91466	1.48951
9.0	2944.31108	4385.56643	1.48951
9.1	3250.17066	4841.14586	1.48951
9.2	3587.85195	5344.15390	1.48951
9.3	3960.66950	5899.53758	1.48951
9.4	4372.28341	6512.53663	1.48951
9.5	4826.73538	7189.44720	1.48951
9.6	5332.48855	7936.01115	1.48951
9.7	5882.45709	8761.09155	1.48951
9.8	6494.12659	9673.03503	1.48951
9.9	7169.46404	10678.93366	1.48951
10.0	7915.12075	11789.61318	1.48951

Table VI. Lanchester-Clifford-Schläfli Functions $F_\alpha(x)$, $H_{1-\alpha}(x)$, and $T_\alpha(x)$ for $\alpha = 3/5$ and x from 1.50 to 10.0.

Table VII. Input Data for Numerical Examples

$\mu = 1, \quad \nu = 2$

$\alpha_0 = 0.06$ X   casualties/minute/Y unit

$\beta_0 = 0.6$ Y   casualties/minute/X unit

$R_\alpha = R_\beta = 2000$ meters

$v = 5$ miles/hour

compute the parameter values shown in Table VIII. We observe from Tables VI and VIII

the predicted agreement between $\Gamma(1-\alpha)/\Gamma(\alpha)$ and the limiting value of $T_\alpha(x)$ as

$x \to +\infty$ (see (52)) for $\alpha = q = 3/5$. We now consider two cases: (I) $R_0 = 2000$

meters, and (II) $R_0 = 1250$ meters.

When $R_0 = 2000$ meters (see Figure 3 of Taylor[21]), we have $K_S = 0$ and

$\tau_0 = 0$. The maximum time that the battle can last is $t_{max} = 14.91$ minutes, since at

this time the attackers reach their final objective (i.e. the defensive position).

We now consider the qualitative behavior of the $\mu = 1$, $\nu = 2$ force-level trajectory

shown in Figure 3 of Taylor[21]. Theorem 2 tells us that X can be annihilated

$\Leftrightarrow x_0/y_0 < 0.420$. By (56) the annihilation time of the X force is given by

$T_q(\tau(t_X^a)) = 3.544\ x_0/y_0$. For $x_0 = 10$, $y_0 = 30$, we have $T_q(\tau_X^a) = 1.18122$ so that

from Table V (using linear interpolation) we obtain $\tau_X^a = 1.009$. Hence, (39) yields

$t_X^a = 14.24$ minutes and $r_X^a = 89.8$ meters. Further results are given in Table IX.

When $R_0 = 1250$ meters (see Figure 3 of Taylor and Brown[24]), we have

$K_S = 5.5923$ minutes, $\tau_0 = 0.0975$, and $t_{max} = 9.32$ minutes. In this case (again,

for $\mu = 1$, $\nu = 2$), X can be annihilated $\Leftrightarrow x_0/y_0 < 0.382$ with [from (54)] the

annihilation time of the X force given by $T_q(\tau_X^a) = (3.565\ u_0 + 0.223)/(0.156\ u_0 + 1.004)$,

where $u_0 = x_0/y_0$. Some further numerical results are given in Table X. Again, these

parametric results should be contrasted with the single $\mu = 1$, $\nu = 2$ force-level

trajectory shown in Figure 3 of reference 24.

8. DISCUSSION

In Section 7 above we have seen how our new definition of power Lanchester

functions (guided by the general requirements for GLF given in Table I) allows one to

conveniently obtain much valuable information about the model (1) with attrition-rate

coefficients (7) without explicitly computing the entire force-level trajectories (see

Note 12). Previously we were limited to only computing force-level trajectories.

Now we can tell who is going to be annihilated and when without explicitly computing

the trajectories (see Note 13). Not only did we answer questions about qualitative

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Table VIII. Parameter Values for Numerical Examples

k_a = 4.0233 × 10^{-3} X casualties/(minute)$^{\mu}$/Y unit

k_b = 2.6979 × 10^{-3} Y casualties/(minute)$^{\nu}$/X unit

p = 2/5, q = 3/5

$\Gamma(p)/\Gamma(q)$ = 1.48951

K_0 = 0

Table IX. Annihilation of the X Force as a Function

of the Initial Force Ratio for R_0 = 2000 meters

(x_0/y_0)	t_X^a (minutes)	r_X^a (meters)
0.333	14.24	89.8
0.250	11.61	443.2
0.200	10.19	633.2

Table X.   Annihilation of the X Force as a Function
of the Initial Force Ratio for R_0 = 1250 meters

(x_0/y_0)	t_X^a(minutes)	r_X^a(meters)
0.333	10.63	_____†
0.250	7.56	235.9
0.200	6.17	422.8

†t_{max} = 9.32 minutes   and x_f = x(r=0) = 1.35.

model behavior (e.g. force annihilation) for specific values of, for example, initial

force levels but also for a range of values of the initial force ratio (i.e. parametric

analysis of model behavior). The results of this paper may be used for other para-

metric analyses (see Note 14), e.g. parametric dependence of battle outcome on attri-

tion-rate coefficients. Thus, our extension of past results[24] allows one to develop

important insights into the dynamics of combat between two homogeneous forces with

temporal variations in fire effectivenesses. With the availability[25,26] of tabula-

tions of the LCS functions, one can now analyze such combat modelled by the power

attrition-rate coefficients (7) with somewhat the same facility as he can for the

constant-coefficient case (see Note 15) and thus aid in parametric analyses.

In his classic 1914 paper[17] Lanchester assumed that the combatants' fire

effectivenesses (as expressed by Lanchester attrition-rate coefficients) were constant

over time and deduced his famous square law

$$\beta\{x_0^2-x^2(t)\} = \alpha\{y_0^2-y^2(t)\}, \tag{57}$$

where α and β denote constant attrition-rate coefficients. It follows from (57)

that (provided there is no "time limit" for the battle)

$$X \text{ will be annihilated} \Leftrightarrow x_0/y_0 < \sqrt{\alpha/\beta}. \tag{58}$$

Thus, we see that equality of Lanchester-type fighting strengths depends on two param-

eters: (I) initial force ratio, and (II) relative effectiveness. When the timing

of military actions is considered, we add a third parameter, the intensity of combat =

$\sqrt{\alpha\beta}$, to this list of significant combat parameters. No such simple relationship like

the square law (57), which yields (58), holds in general for variable attrition-rate

coefficients when $a(t)/b(t) \neq$ constant. However, by transforming the independent

variable t to normalize the battle's time scale by the intensity of combat, we found

(see equation (37)) that the course of such variable-coefficient combat depends on

only two weapon-system parameters: (I) relative fire effectiveness, $R(t) = b(t)/a(t)$,

and (II) intensity of combat, $I(t) = \sqrt{a(t)b(t)}$. Moreover, we extended (58) to combat

modelled with the power attrition-rate coefficients with "no offset" (7) (see Theorem 2).
This is the _first_ time that such a generalization of the square law has been obtained
for the variable-coefficient Lanchester-type model (1) with $a(t)/b(t) \neq$ constant. We
observe that for $K_S > 0$ this "exact" outcome-prediction relation (i.e. necessary
and sufficient condition for force annihilation) involves higher transcendental functions
(here, the LCS functions) and is complementary to the sufficient condition (involving
only elementary functions) given by Taylor and Parry[28] for $K_S > 0$.

Work by BONDER[5,7], Clark[12], and others[2,9] on the prediction of Lanchester
attrition-rate coefficients (see Taylor and Brown[24] for further discussion and
references) has generated interest in variable-coefficient Lanchester-type models.
Interest in the power attrition-rate coefficients with "no offset" (7) is provided by
S. Bonder's[4,6] model (5) and his examination of predicted attrition-rate for various
weapon systems (see pp. 196-200 of reference 9). However useful our results may be
in their own right, they have far greater import: (I) they are a model for the treat-
ment of other Lanchester functions and their tabulations, and (II) they may be used in
the numerical determination of the parity-condition parameter[27] Q^* for related
attrition-rate coefficients (for example, (4) with $K_O > 0$). In the future we will
show how our tabulations of the LCS functions play a key role in the numerical deter-
mination of the parity-condition parameter Q^* for the general power attrition-rate
coefficients (4) with positive "offset" (i.e. $K_O > 0$).

We have extended our mathematical theory[24] of variable-coefficient Lanchester-
type equations of "modern warfare" for combat between two homogeneous forces in order
to be able to more thoroughly analyze such models. The classic ordinary differential
equation theories (see, for example, HILLE[15]) were inadequate to supply all the
answers sought about such combat models (cf. questions (Q1)-(Q4) in Section 1 above).
The mathematical theory of the model (1) with coefficients (7) is now nearly as complete
as that of the constant-coefficient model. Such results as we have given here are very
useful for understanding the dynamics of combat, i.e. how the trading of casualties

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will be projected over time. H. K. WEISS[30] has emphasized that such a simplified model of a combat situation is particularly valuable when it leads to a clearer understanding of significant relationships that would tend to be obscured in a more complex model. As is always the case, however, the insights geined into combat dynamics are no more valid than the models themselves.

9. SUMMARY

In this paper we have introduced new mathematical functions (Lanchester-Clifford-Schläfli, or LCS, functions) that allow important information (in particular, force-annihilation prediction) to be obtained without explicitly computing force-level trajectories for the variable-coefficient Lanchester-type model (1) with power attrition-rate coefficients with "no offset" (7). Our development was based on new theoretical considerations: we gave a new general discussion of representing the solutions to the X and Y force-level equations in terms of general Lanchester functions (GLF) and gave the general properties that these GLF should possess; we showed that there are essentially only two kinds of GLF that satisfy these requirements (exponential-like GLF and hyperbolic-like GLF) and that the hyperbolic-like functions are to be preferred. Moreover, the exponential-like GLF are an essential theoretical construct, since they play a key role in determining force-annihilation-prediction conditions (i.e. showing that the reciprocal of the parity condition parameter is equal to the limiting value of the quotient of two hyperbolic-like general Lanchester X-functions). We stressed that such building blocks should be chosen to yield as much information as possible about the model (and as conveniently as possible). We saw that the analysis of, for example, the X force-level equation was facilitated by transforming the battle's time scale and that the only two weapon-system parameters affecting the course of combat are the relative fire effectiveness and the intensity of combat. These results extended and unified our mathematical theory of variable-coefficient Lanchester-type equations of "modern warfare" (see reference 24).

We then applied our general mathematical theory to the special case of combat modelled by power attrition-rate coefficients with "no offset." Our new definition of Lanchester-Clifford-Schläfli (LCS) functions was required for these power attrition-rate coefficients in order to answer questions about battle outcome without explicitly computing force-level trajectories (i.e. to predict battle outcome/force annihilation). The mathematical theory of this variable-coefficient Lanchester-type models of "modern warfare" (modelling, for example, weapon systems with the same effective range) is now nearly as complete as that of the constant-coefficient model. With tabulations of the new LCS functions now available, one can study this variable-coefficient model almost as easily and thoroughly as Lanchester's classic constant-coefficient model.

NOTES

1. Following terminology introduced in reference 24, we will refer to Lanchester functions corresponding to the power attrition-rate coefficients (4) with $K_0 > 0$ as offset power Lanchester functions (see Section 1). The power Lanchester (i.e. LCS) functions correspond to $K_0 = 0$, i.e. to the power attrition-rate coefficients with "no offset" (7).

2. The equations (1) are only valid for $x,y > 0$. The first, for example, becomes $dx/dt = 0$ for $x = 0$.

3. Further information on sets of circumstances that have been hypothesized to yield the combat equations (1) (with constant coefficients) may be found in BRACKNEY[11] and Weiss[29].

4. It is impossible for all $a(t),b(t) > 0$ satisfying Condition (A) to have $x_i,y_i > 0$ for $i = 1,2$ and all $t > t_0$ such that $\dot{x}_i = -\sqrt{k_b/k_a}\, a(t)y_i$ and $\dot{y}_i = -\sqrt{k_a/k_b}\, b(t)x_i$ and x_1,x_2 are linearly independent. If it were possible, then $|S(t)| = -x_1y_2 + x_2y_1 = C_S$ and without loss of generality we may take $C_S > 0$.

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Introducing $u_i = x_i/y_i$ and $\Delta = u_2 - u_1$, we would have $|S(t)| = y_1 y_2 \Delta = C_S > 0$ and $\dot{\Delta} = \sqrt{k_b/k_a}\, b(t)(u_1+u_2)\Delta > \sqrt{k_b/k_a}\, b(t)\Delta^2 > 0$ for $t > t_0$ so that $\Delta(t) > 0$ is strictly increasing for $t \geq t_0$. It would follow that $1/\Delta(t_0) - 1/\Delta(t) \geq \sqrt{k_b/k_a} \int_{t_0}^{t} b(s)ds$, which is impossible for $b(t)$ such that $\lim_{t \to +\infty} \int_{t_0}^{t} b(s)ds = +\infty$.

5. To keep x_1, x_2, y_1, and y_2 as "simple" as possible we specify that their initial values at t_0 to be either 0 or 1. In order that $|S(t)| = x_1 y_2 - x_2 y_1 \neq 0$, we must therefore have either $x_1(t=t_0) = y_2(t=t_0) = 0$ and $x_2(t=t_0) = y_1(t=t_0) = 1$, or $x_1(t-t_0) = y_2(t=t_0) = 1$ and $x_2(t=t_0) = y_1(t=t_0) = 0$. We consider the first possibility with similar arguments holding for the second. In this first case examination of the differential equations with initial conditions shows us that $x_1 = S_X$, $x_2 = C_X$, $y_1 = C_Y$, and $y_2 = S_Y$, i.e. the functions coincide with the hyperbolic-like GLF of Table III. Thus, we need not consider $L = I$, since the same results may be obtained by using another one of Lemma 1's feasible forms for L.

6. We conjecture that some condition like $\lim_{t \to +\infty} \tau(t) = +\infty$ is sufficient to guarantee that Q^* is unique.

7. This intuitively obvious result may be proved by observing the identity

$$\int_{s}^{t} \{b(\sigma)x^2(\sigma) + a(\sigma)y^2(\sigma)\}d\sigma = x(s)y(s) - x(t)y(t).$$

A less obvious fact is that unless at least one of $\int_{0}^{\infty} a(t)dt$ and $\int_{0}^{\infty} b(t)dt$ is unbounded, then neither $x(t)$ nor $y(t)$ need ever be annihilated (see Hille[15]). As an example of this situation, consider the battle with attrition-rate coefficients (7), $K_S > 0$, and $\mu = \nu < -1$. Then $x(t) = x_0 \cosh \theta(t) - y_0 \sqrt{k_a/k_b} \sinh \theta(t)$, where $\theta(t) = \{-1/(\nu+1)\}\{1/K_S^{-(\nu+1)} - 1/(t+K_S)^{-(\nu+1)}$. Let $\theta_\infty = \{-1/(\nu+1)\}\{1/K_S^{-(\nu+1)}\} > 0$ and finite. It follows that $\nu < -1$ and $K_S > 0$ can be chosen so that even though $x_0 < y_0\sqrt{k_a/k_b}$ we have $\lim_{t \to +\infty} x(t) = x_0 \cosh \theta_\infty - y_0\sqrt{k_a/k_b} \sinh \theta_\infty > 0$.

8. In general, the value of $Q*$ will not be known exactly. Unfortunately, errors in the initial conditions for the exponential-like GLF become exponentially magnified over time. The situation is even worse for $\eta_E(t;Q*) = E_{2X}^+(t;Q*) = E_X^+(t;Q*)/E_X^-(t;Q*)$, which is used to determine the time that the X force will be annihilated.

9. The term _quasi-autonomous_ was coined by Taylor[22] (_see_ also TAYLOR[23]) to denote a system of differential equations that may be transformed to an autonomous system (_see_ for example, p. 163 of PETROVSKI[18]) by a change of the time scale. Special cases of such Lanchester-type equations have been considered by, for example, Farrell[9]. and TAYLOR[20]. More general (possibly nonlinear) quasi-autonomous Lanchester-type equations have been studied by Taylor[22],[23] (_see_ also Note 4 of Taylor and Brown[24]).

10. If we were to specify the initial conditions of the GLF at $t = 0$ instead of $t = t_0$, then (32) would reduce to $x(t) = x_0 \widetilde{C}_X(t) - y_0 \sqrt{k_a/k_b}\ \widetilde{S}_X(t)$. However, when the initial conditions for the hyperbolic-like GLF are not given at t_0, a separate tabulation of, for example, $\widetilde{C}_X(t)$ must be used for each different value of t_0 (i.e. $\widetilde{C}_X = \widetilde{C}_X(t;t_0)$).

11. Although the solution to the X force-level equation (2) with the power attrition-rate coefficients (7) may be expressed in terms of known higher transcendental functions (_see_ Taylor[21], Taylor and Brown[24], and Taylor and Comstock[27]), we have chosen to introduce the LCS functions, since tabulations of these other functions are not available for the full range of parameter values of interest in Lanchester combat theory. For example, we can construct such solutions with modified Bessel functions of the first kind of fractional order, but tabulations of these (_see_, for example, Abramowitz and Stegun[1]) only exist for a restrictive set of values of the order p (i.e. $p = \pm 1/4, \pm 1/3, \pm 1/2, \pm 2/3, \pm 3/4$), where $p = (\mu+1)/(\mu+\nu+2)$. Furthermore, tabulations of functions corresponding to the quotient of, for example, two GLXF do not apparently exist at all. Consequently, we have introduced our new LCS functions, which provide much of the information desired about such battles. The naming of our LCS functions follows from the facts that a function similar to $F_\alpha(x)$ was introduced

by LUDWIG SCHLÄFLI[19] (1814-1895) in 1867, while a related one appears in a posthumous fragment of the great English geometer William Kingdon Clifford (1845-1879) (see pp. 343-348 of CLIFFORD[13]).

12. In his well-known survey paper on the Lanchester theory of combat, Dolansky[14] suggested the development of outcome predicting relations without solving in detail and/or computing force-level trajectories as one of several problems for future research. Our Theorem 2 is a step towards this problem's resolution (see also references 22, 27, and 28).

13. Bonder and Honig[10] point out, however, that force annihilation may not be the appropriate criterion for evaluating many military operations, especially when force annihilation does not occur. See pp. 192-242 of Bonder and Farrell[9] for a detailed Lanchester-type analysis of an attack situation for which other "end of battle conditions" play the major role in the evaluation process. Nevertheless, it is of interest to know when and why force annihilation will occur.

14. S. BONDER[8] has suggested that an increased emphasis be placed on parametric analyses in systems analysis studies (see pp. 21-22 of reference 8).

15. One significant exception is that the outcome of fixed-force-level-breakpoint battles (for example, Y "wins" when $x_f = x(t_f) = x_{BP}$ but $y_f > y_{BP}$, where t_f, x_f, y_f denote final values and x_{BP} denotes X's breakpoint) with $x_{BP}, y_{BP} > 0$ and $a(t)/b(t) \neq$ constant cannot apparently be analyzed in the manner described in this paper (see Taylor and Comstock[27]).

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